Methods section for ECMWF-led paper on precipitation statistical postprocessing

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1 Quantile mapping

The statistical adjustment of ensemble forecasts begins with quantile mapping. Assume we have a raw ensemble forecast amount $\tilde{x}$ which provides an estimate of the true (unknown) precipitation amount x. Assume we have cumulative distribution functions (CDFs) for the forecast and analyzed $\Phi_f(x)$ and $\Phi_a(x)$ respectively. Given a precipitation amount, the CDFs return the non-exceedance probability q. The inverse function, $\Phi_a^{-1}(q)$ is the quantile function, which returns the corresponding analyzed amount associated with that quantile. Quantile mapping thus adjusts the forecast to be consistent with the analyzed CDF:

$$\tilde{y} = \Phi_a^{-1}\left[\Phi_f(\tilde{x})\right] . \tag{1}$$

CDFs are needed to perform the quantile mapping. In this application, CDFs were generated separately for each lead time and each model grid point using 1998-2017 m=11 member reforecast training data and using the nine reforecast dates of the year closest to the Julian day of the real-time forecast. Following Hamill and Scheuerer (2018; hereafter HS18), **ns** supplemental locations were used to provide extra training data. [****Esti, verify this ****] The supplemental locations were selected based on the similarities of analyzed climatologies and terrain characteristics, directly following HS18.

Quantile mapping will be applied during both training and real-time forecasting steps. First, during the training phase, it will be applied to the reforecast data in a cross-validated manner to ameliorate systematic errors such as the over-forecasting of light precipitation amounts. With 20 years of reforecast data, 19 years of data are used to populate the CDFs, and the remaining year of training data is then quantile mapped. The procedure is repeated to provide quantile-mapped precipitation amounts spanning the 20 years. These data are then used in a second step of the training process, as input for developing *closest-member histograms*, discussed below.

Quantile mapping is also applied to the real-time ensemble as the first step in the correction of systematic error. In this case, the CDFs for the quantile mapping were developed from the full 20 years × 11 members × 9 cases × ns supplemental locations, thus providing XXXX total samples to generate the empirical CDF.

Because of the model's tendency to over-forecast light precipitation, quantile mapping sometimes adjusts a forecast light precipitation amount to zero. Suppose the CDFs indicated an under-forecasting of light precipitation. In this case there are multiple quantiles of $\Phi_f(x)$ that likely are associated with zero, and we face a non-uniqueness problem: given a zero forecast amount to quantile map, is this representing the 0th percentile of the forecast CDF, or perhaps the 5th percentile? This problem is avoided by instituting an ad-hoc rule, such that zero forecast amounts are retained without quantile mapping.

2 Generating weights for sorted ensemble members

The second corrective step during the training process is applied after the quantile mapping of ensemble members has occurred. Suppose for the moment there is a rational basis to believe that the analyzed state is more likely to be near some sorted members than others. Let's assume we have a vector of weights $\mathbf{w} = [w_{(1)}, \ldots, w_{(m)}]$ associated with the sorted members that reflect this likelihood, where $_{(i)}$ denotes the ith rank. Weighted probabilities can then be generated in a straightforward manner. When considering the probability of exceeding the threshold amount t, we define an indicator function for the ith sorted member:

$$I(i) = \begin{cases} 0 & \text{if } \tilde{y}_{(i)} < t \\ 1 & \text{if } \tilde{y}_{(i)} \geq t \,. \end{cases} \tag{2}$$

Weighted probabilities of exceeding the amount t are then generated as follows:

$$P(x > t) = \sum_{i=1}^{m} I(i)\ w_{(i)} \,. \tag{3}$$

The question then turns to how to objectively generate weights associated with each sorted member. A procedure for doing so was described in HS18 using the previously mentioned closest-member histograms. To generate closest-member histograms, after a set of cases of ensemble training data for a particular lead time is quantile mapped, we have an 11-dimensional vector $\tilde{\mathbf{y}} = [\tilde{y}_1, \ldots, \tilde{y}_{11}]$ of reforecast quantile-mapped estimates of the unknown precipitation amount. These data are sorted, $\tilde{\mathbf{y}}^s = [\tilde{y}_{(1)}, \ldots, \tilde{y}_{(11)}]$, and then they are compared to the analyzed precipitation amount. Closest-member histograms are generated tallying over many samples which sorted member is closest to the analyzed amount by rank. Following HS18, separate closest-member histograms are generated in this application for different quantile-mapped ensemble-mean amounts.

A complication arises in the application of this methodology to ECMWF real-time forecasts. For ECMWF, the training data consisted of 11 reforecast members, resulting in 11-dimensional closest-member histograms. The real-time ensemble currently is comprised of 51 members. Consequently, the closest-member histograms cannot be used directly given the differing ensemble sizes. However, a straightforward procedure can be applied to generate closest-member histograms for the 51-member ensemble through the use of Beta distributions (Wilks 2011, section 4.4.4). A Beta distribution provides a continuous probability densities associated here with a quantile q in the range of (0,1). The pdf $f(q,\alpha,\beta)$ of the Beta distribution is

$$f(q,\alpha,\beta) = \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right) q^{\alpha-1}(1-q)^{\beta-1};\quad 0<q<1;\quad \alpha,\beta>0. \tag{4}$$

α and β are the parameters of the Beta distribution and $\Gamma(\cdot)$ is the Gamma function. Parameter estimates $\hat{\alpha}$ and $\hat{\beta}$ are commonly generated from the method of moments as

$$\hat{\alpha} = \frac{\bar{q}^2(1-\bar{q})}{s^2} - \bar{q} \tag{5}$$

and

$$\hat{\beta} = \frac{\hat{\alpha}(1-\bar{q})}{\bar{q}}, \tag{6}$$

where $\bar{q}$ and s^2 are the sample mean and standard deviation, respectively. Beta distributions have flexible shapes and can be fit to resemble the closest-member histograms. An example of this is provided in Figure 1, illustrating closest-member histograms and histograms re-generated from the corresponding integration of fitted Beta distributions.

The procedure for generating closest-member histograms for the real-time, 51-member ensemble is as follows: (a) fit a Beta distribution to the 11-dimensional closest-member histogram based on the ECMWF reforecast training data; (b) generate the weights associated with the larger m=51-member ensemble by integrating the Beta distribution into 51 equally spaced regions spanning 0 to 1. For step (a), sample means and variances are needed to apply the method of moments to estimate the Beta distribution parameters. Let $\mathbf{w}^{11}$ represent the appropriate 11-dimensional closest-histogram vector of weights from the reforecast ensemble based on the quantile-mapped ensemble mean. Let's also denote a vector $\mathbf{a}$ that provides the corresponding central value associated with each rank in the closest-member histogram when mapped to the interval (0,1):

$$\mathbf{a} = \left(a_1,\ldots,a_{11}\right) = \left(\frac{0}{11}+\frac{1}{2\times 11},\ldots,\frac{10}{11}+\frac{1}{2\times 11}\right) \tag{7}$$

The sample mean $\bar{q}$ is

$$\bar{q} = \sum_{i=1}^{11} a_i w_i^{11}, \tag{8}$$

and the sample variance is calculated from a closest-histogram weighted sum of squared differences from the sample mean:

$$s^2 = \frac{10}{11} \sum_{i=1}^{11} (a_i - \bar{q})^2 \times w_i^{11} \quad . \tag{9}$$

The second step is generating the closest-member histogram weights through integration of the fitted Beta distribution. Let j indicate the rank in the sorted, 51-member ensemble and the index in the closest-member histogram vector $\mathbf{w}^{51}$. The closest-member histogram weight for this rank is calculated as

$$w_j^{51} = \int_{(j-1)/m}^{j/m} f(q, \hat{\alpha}, \hat{\beta}) dx \, . \tag{10}$$

Figure 2 provides an example of the 51-dimensional closest-member histograms that have been recreated from the 11-dimensional data illustrated in Fig. 1.

With the 51-dimensional closest-member histograms generated from the training data, the statistical adjustment of the real-time forecasts can proceed. Real-time forecasts are quantile-mapped using reforecast-based CDFs (eq. 1). Based on the ensemble-mean precipitation amount, the appropriate closest-member histogram is selected, and probabilities are generated using eqs. (2) and (3).

3 Generating synthetic, equally likely ensemble members leveraging closest-member histogram weights

The post-processed ensemble precipitation guidance is to be used as a forcing to an ensemble hydrologic prediction system to produce ensemble of streamflow predictions. With the procedure described thus far, we have precipitation event probabilities and the data used to generate them, an ensemble of quantile-mapped members and closest-member histogram weightings. We do not have an ensemble of exchangeable, equally likely members; we have weighted members instead.

This introduces a challenge for ensemble streamflow prediction, where a presumed ensemble of exchangeable, equally likely members is assumed. Could one produce probabilistic realistic streamflow forecasts by weighting the resulting ensembles of streamflow simulations with the closest-member histogram weights? Unfortunately, since the closest-member histogram weights will vary with forecast lead time and with ensemble-mean precipitation amount, there is no one unique vector of weights that can be applied to a streamflow time series, even at one single grid point. Further, the streamflow at a point represents not only rainfall at that point but at grid points which drain into that point. Closest-member histogram weighting of ensemble streamflow output is thus inappropriate.

We now describe a procedure that generates modest adjustments to the quantile-mapped ensemble to make the data more exchangeable in character while preserving rank structure. The adjustment procedure leverages only the data at hand, the quantile-mapped ensemble for a particular lead time and grid point and the associated 51-dimensional closest-member histogram. The procedure to be applied to adjust the quantile-mapped members again leverages the machinery of quantile mapping, using it to perform a stretching of the original ensemble so that members are more equally likely in their statistical character. It is very similar to the ensemble copula coupling (transformation) or ECC-T method described in Schefzik et al. (2013).

For the procedure here to adjust the quantile-mapped members to have characteristics more like equally likely members, $\Phi_f(x)$ will no longer represent a CDF of past forecasts. Instead, it now depicts a distribution for a particular grid point fitted to today's quantile-mapped under the assumption that all members are given equal weight. Hereafter this distribution is referred to as the "prior". Similarly, $\Phi_a(x)$ now depicts a distribution for a particular grid point fitted to today's quantile-mapped and closest-histogram weighted ensemble, which we denote as the "posterior".

How are the prior and posterior distributions estimated? In both cases we estimate the parameters of a distribution with a point mass at zero, known as the fraction zero, or FZ, and a fitted Gamma distribution (Wilks 2011, section 4.4.3) for the positive amounts.

The procedure for estimating the fitted distributions for the prior (quantile-mapped) and posterior (quantile mapped and weighted) are functionally equivalent. In the latter case, weights in the procedure are supplied by the closest-member histograms. In the former, weights are constant, 1/51. Again, assume the sorted, quantile-mapped ensemble is available: $\tilde{\mathbf{y}}^s = [\tilde{y}_{(1)}, \ldots, \tilde{y}_{(m)}]$, and assume a set of weights have been determined: $\mathbf{w} = [w_{(1)}, \ldots, w_{(m)}]$, equally weighted or closest-member histogram weighted. To determine the estimated fraction zero, we generate a 51-member indicator function for whether the precipitation is effectively non-zero or not. The threshold for such a determination is 0.01 mm. The indicator function for the ith member is

$$I(i) = \begin{cases} 0 & \text{if } \tilde{y}_{(i)} < 0.01mm \\ 1 & \text{if } \tilde{y}_{(i)} \geq 0.01mm \end{cases} \tag{11}$$

The estimated fraction zero is generated from the weighted sum of the indicator functions:

$$\hat{FZ} = 1. - \sum_{(i)=1}^{51} I(i)\, w_{(i)} \;. \tag{12}$$

Suppose of the 51 sorted, quantile-mapped members there were n remaining samples with positive precipitation. We then generate weighted estimated of mean of positive quantile-mapped forecast values:

$$\bar{\tilde{\mathbf{y}}}_w^+ = \frac{\sum_{(i)=51-n}^{51} \tilde{y}_{(i)}^+ w_{(i)}}{\sum_{(i)=51-n}^{51} w_{(i)}} \tag{13}$$

Following the procedure outlined in Wilks (ibid) and using the Thom (1958) estimator, we estimate a parameter D_w:

$$D_w = ln(\bar{\tilde{\mathbf{y}}}_w^+) - \sum_{(i)=51-n}^{51} ln\left[\tilde{y}_{(i)}^+ w_{(i)}\right] \ . \tag{14}$$

From this it is now possible to estimate the shape α and scale β parameters of the Gamma distribution:

$$\hat{\alpha} = \frac{1+\sqrt{1+4D_w/3}}{4D_w} \ , \tag{15}$$

$$\hat{\beta} = \frac{\bar{\tilde{\mathbf{y}}}_w^+}{\hat{\alpha}} \ . \tag{16}$$

With the fraction zero and gamma-distribution parameters separately estimated for quantile-mapped unweighted and weighted ensembles, we have fitted $\Phi_f(x)$ and $\Phi_a(x)$ and the original ensemble of quantile-mapped values. The mapping procedure from eq. (1) is now applied. This procedure is illustrated in Fig. 3. Panel (a) shows the empirical CDF (horizontal black lines) and the $\Phi_f(x)$ (red line) for a particular grid point. Panel (b) shows the weighted empirical (horizontal black lines) and $\Phi_a(x)$ for the closest-member histogram weighted ensemble. Finally, panel (c) illustrates the quantile-mapping procedure, where the original quantile-mapped values are shifted slightly (the horizontal black arrows) consistent with the differences in prior and posterior distributions. Figure 4 shows the effect of this for a sample grid point. The ensemble after application of this process is more stretched out, with higher high precipitation amounts and lower low precipitation amounts. Figure 5 shows before vs. after scatterplots of precipitation data for two grid points separated by approximately 422 km. As shown, the structure of relationships between data at the two grid points is preserved after the stretching of the data.

4 References

Hamill, T. M., and M. Scheuerer, 2018: Probabilistic precipitation forecast postprocessing using quantile mapping and rank-weighted best-member dressing. *Mon. Wea. Rev.*, accepted pending minor revision.

Schefzik, R.,Thorarinsdottir, T. L., and Gneiting, T: 2013: Uncertainty quantification in complex simulation models using ensemble copula coupling. *Statist. Sci.*, **28**, 616-640. doi:10.1214/13-STS443.

Thom, H. C. S., 1958: A note on the Gamma distribution. *Mon. Wea. Rev.*, **86**, 117-122.

Wilks, D. S., 2011. *Statistical Methods in the Atmospheric Sciences (3rd Ed).* Academic Press, 676 pp.

5 Figures

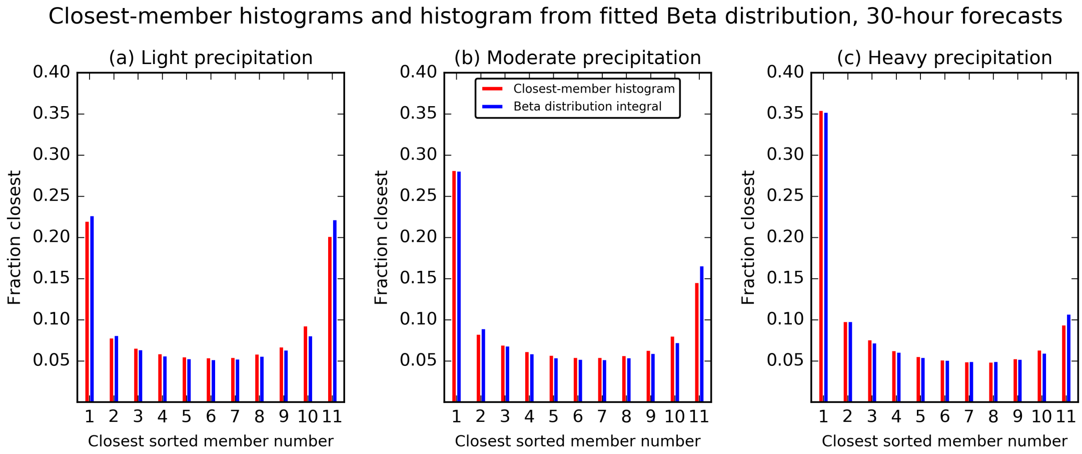

Figure 1: Illustration of 11-dimensional closest-member histograms generated for the ECMWF reforecast ensemble using 9 reforecast dates centered on 14 July. Forecasts are of accumulated precipitation from 6 to 30 h, and analyzed data was extracted from the EFAS precipitation analysis database. Fitted Beta distributions are also shown alongside the closest-member histograms. Following HS18, histograms were generated separately for different ensemble-mean amounts. Quantile-mapped ensemble-mean precipitation was classified as light if the mean is ≥ 0.01 mm and < 2.0 mm. It is classified as moderate if the mean is ≥ 2.0 mm and < 6.0 mm. Finally, it is classified as heavy if the mean is ≥ 6.0 mm.

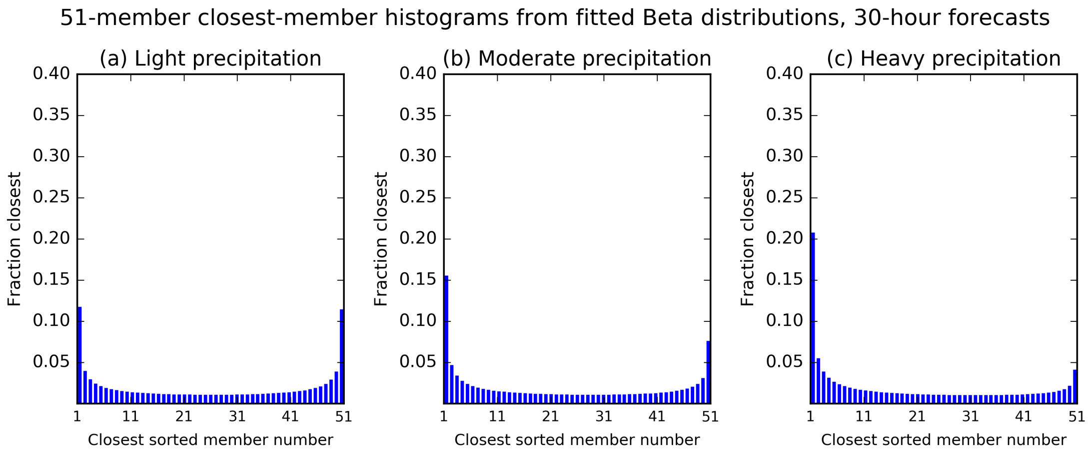

Figure 2: 51-dimensional closest-member histograms generated through a Beta-distribution fitting procedure using the data from Fig. 1.

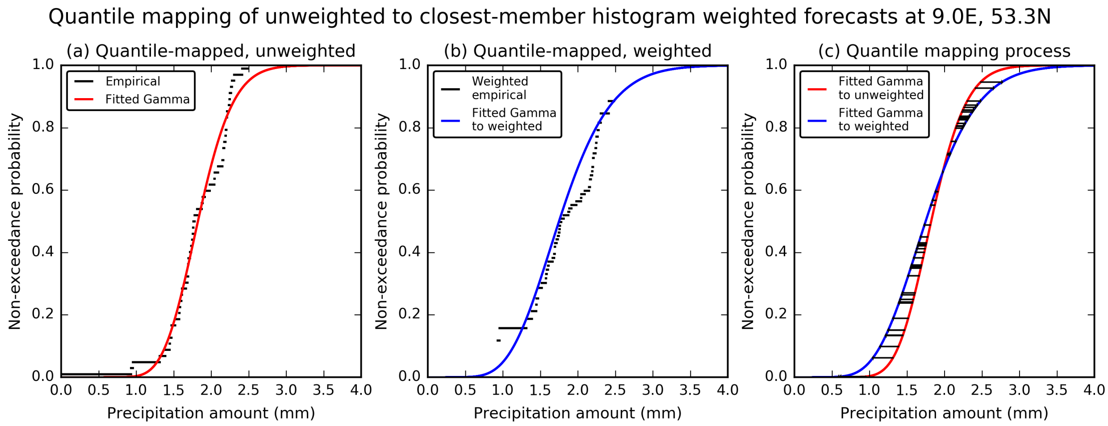

Figure 3: Illustration of the procedure for adjustment of quantile-mapped precipitation forecasts. Data in this example were accumulated 6- to 30-hour forecasts of accumulated precipitation for the forecast initialized at 00 UTC 14 July 2016 for the model grid point nearest to 9^oE, 53.3^oN. (a) Empirical distribution (black) and fitted Gamma prior distribution (red) for the quantile-mapped, unweighted ensemble. (b) Empirical (black) and fitted posterior Gamma distribution (blue) for the closest-member histogram weighted ensemble. (c) Illustration of the mapping procedure; original quantile-mapped values are shifted slightly (the horizontal black arrows) consistent with the differences in prior and posterior distributions.

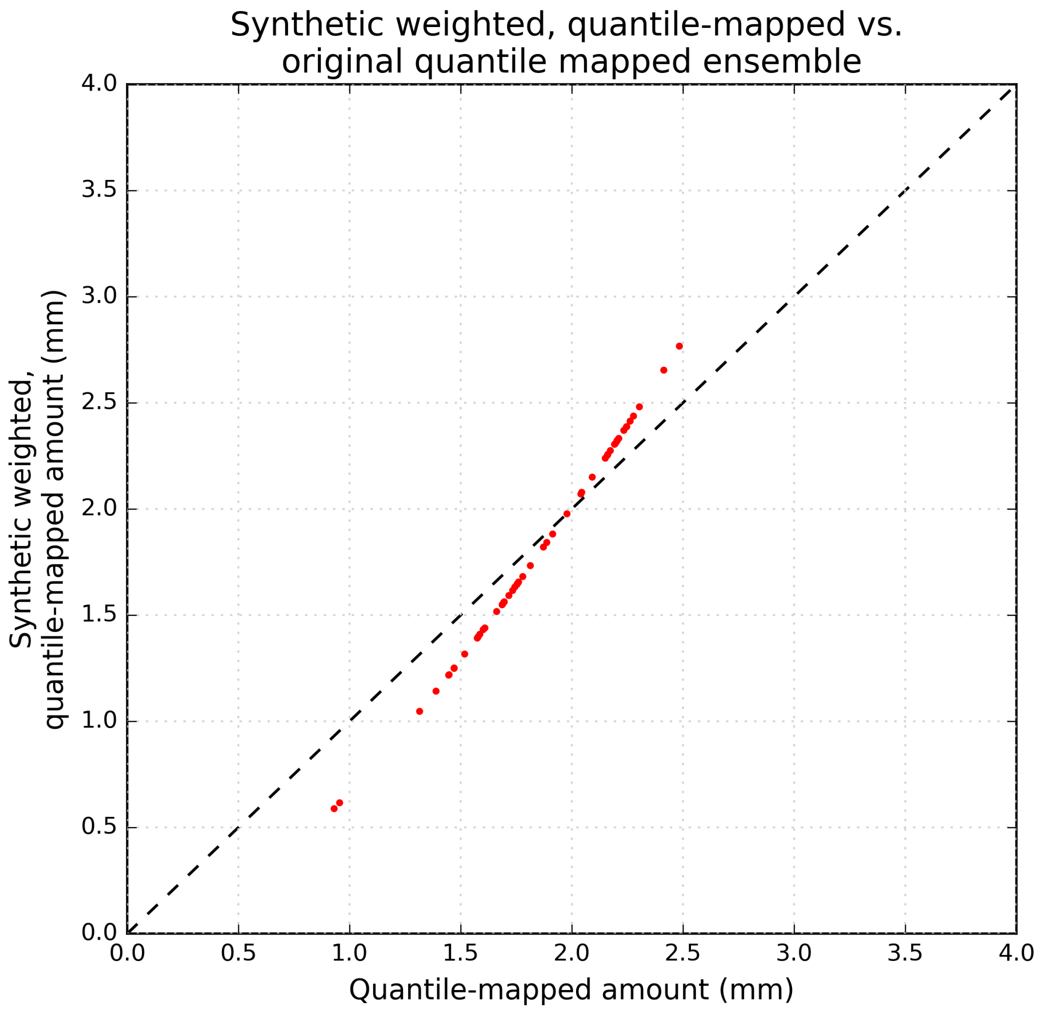

Figure 4: Illustration of the effective stretching of the ensemble forecasts through the mapping procedure using the ensemble forecast data of Fig. 3. Abscissa provides the original quantile-mapped ensemble, and values on the ordinate provide the values after adjustment.

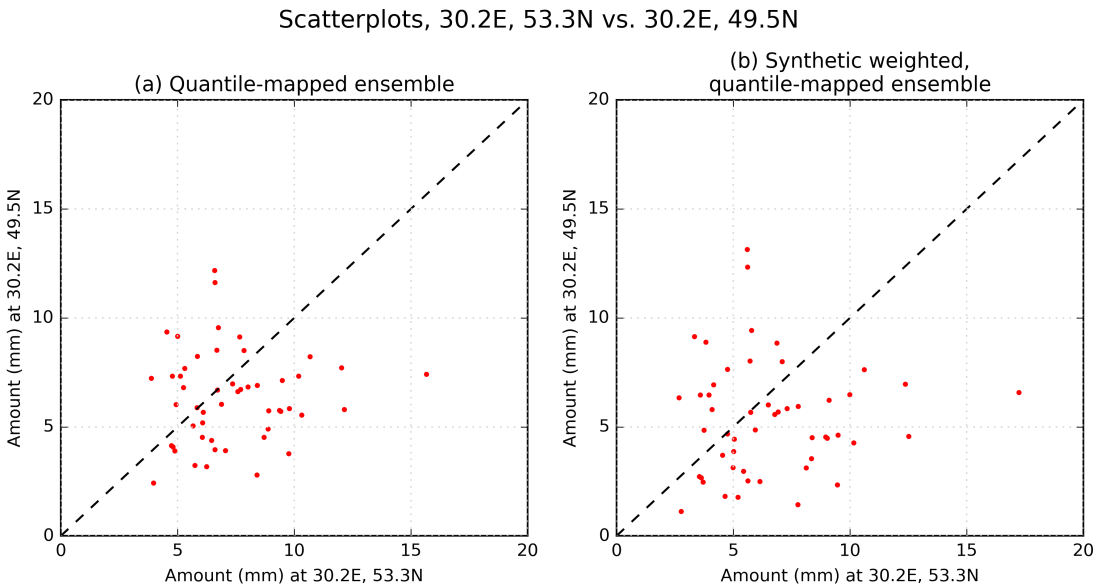

Figure 5: Relationship between 6- to 30-h precipitation forecasts initialized at 00 UTC 14 July 2016 for two grid points (locations noted in the plot). (a) Precipitation scatterplot before adjustment, and (b) after adjustment.